JASS

Rice's integrals – a method for solving generalised diffrences

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Based on an article of Philippe Flajolet and Robert Sedgewick:

Mellin transforms and asymptotics: finite differences and Rice's integrals

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1 Basics of complex analysis

1.1 Complex differentiability

Complex mapping on an open set E

$$f : E \subset \mathbb{C} \to \mathbb{C}, z = x + yi \mapsto f(z) = u(x, y) + iv(x, y) \tag{1}$$

Continuity like in 2 dimensions

$$g : E' \subset \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (u(x, y), v(x, y)) \tag{2}$$

Differentiability of g at (x_0, y_0):

$$A := \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}_{x=x_0, y=y_0} \tag{3}$$

A is a linear mapping with

$$\forall (x,y) \in E' : g(x,y) = g(x_0, y_0) + A \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + B(x - x_0, y - y_0) \tag{4}$$

where B is a mapping with $B(0,0) = (0,0)^T$ and $\lim_{(x,y)\to(0,0)} \frac{B(x,y)}{|(x,y)|} = (0,0)^T$

This is an extension of

$$h'(x_0) = \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} \tag{5}$$

Linear mappings $E \subset \mathbb{C} \longrightarrow \mathbb{C}$ are complex multiplication:

$$\forall z \in E : f(z) = f(z_0) + f'(z_0) \cdot ((x - x_0) + i(y - y_0)) + b((x - x_0) + i(y - y_0)) \tag{6}$$

where b is a complex mapping with $b(0) = 0$ and $\lim_{z \longrightarrow 0} \frac{b(z)}{|z|} = 0$.

If such a number $f'(z_0)$ exists, f is said to be complex differatiable at z_0.

Matrix-vector-multiplication vs. the product of two complex numbers:

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 \\ a_{2,1}x_1 + a_{2,2}x_2 \end{pmatrix} \tag{7}$$

$$(a_1 + ia_2)(x_1 + ix_2) = (a_1x_1 - a_2x_2) + i(a_2x_1 + a_1x_2) \tag{8}$$

correspondence $a_{1,1} = a_1 = a_{2,2}$ and $a_{1,2} = -a_2 = a_{2,1}$

A real function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ understood as a complex function is complex differentiable, iff $u_x = v_y$ and $u_y = -v_x$(Cauchy-Riemann Differential Equation[a]).

complex differentiable function $\Rightarrow$ differentiable in a real sense

real differentiable function $\wedge$ CRDEs $\Rightarrow$ complex differentiability

[a] further abreaviated with CRDE

Definition 1.1 *A complex mapping f as introduced in (1) is said to be differentiable at a point $z_0 \in E$ with derivative $f'(z_0)$, iff (6) holdes for some appropriate b.*

Definition 1.2 *The complex mapping f is said to be holomorphic at a point z_0 on E, iff f is complex differentiable at any point of a neighbourhood $F \subset E$ of z_0, which is more than just complex differentiable at z_0.*

Definition 1.3 *The complex mapping f is said to be holomorphic on an open set $F \subset E$, iff f is holomorphic at any point $z_0 \in F$.*

Examples of complex derivativs:
$(z^n)' = nz^{n-1}$, $(e^z)' = e^z$ or $(\sin(z))' = \cos(z)$

1.2 Integration

Real Analysis: Integration over intervalls; integral is limit of sums respecting discretisations, which get "finer":

$$\int_a^b f(x)dx = \lim_{n\to\infty,|Z_n|\to 0} \sum_{k=0,x_{n,k}\in[z_{n,k},z_{n,k+1}]}^{n} f(x_{n,k})(z_{n,k+1}-z_{n,k}) \tag{9}$$

where $z_{n,0} = a < z_{n,1} < \ldots < z_{n,n} < z_{n,n+1} = b$ for every n

"Speed" on the intervall is not important:

$$\int_a^b g(\gamma(s))\gamma'(s)ds = \int_c^d g(t)dt \tag{10}$$

where $\gamma : [a,b] \to [c,d]$ is piecewise differentable and monotonous and $g : [c,d] \to \mathbb{R}$ is piecewise continous.

Integrate "along" a piecewise differentiable curve $\gamma : \mathbb{R} \to \mathbb{C}; t \mapsto \gamma(t)$.

γ is called an integration path.

Definition 1.4 *The integral of a complex function f along a curve*[a] γ *is:*

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt = \int_a^b \Re\left(f(\gamma(t)) \cdot \gamma'(t)\right)dt + i\int_a^b \Im\left(f(\gamma(t)) \cdot \gamma'(t)\right)dt \quad (11)$$

[a] Note, that we integrate along a curve in the first place and not the image of the curve.

Example: $\gamma : [0, 2\pi] \to \mathbb{C}; t \mapsto e^{it}$ and $f(z) = \frac{1}{z}$.

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it} dt = \int_0^{2\pi} idt = 2\pi i \tag{12}$$

The length of a path of integration is

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{(\Re\gamma'(t))^2 + (\Im\gamma'(t))^2} dt \tag{13}$$

For a piecewise continous $f : [a, b] \to \mathbb{C}$:

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|\, dt \tag{14}$$

For f, continous on the image of a path of integration γ and M the maximum absolute value of f on that image, we have:

$$\left| \int_\gamma f(t)dt \right| \leq M \cdot \Lambda(\gamma) \tag{15}$$

A surjective, piecewise continous differentiable, real function ϕ is a parameter transform, iff for all points of its domain, where ϕ is differentiable, the derivative is greater than 0.

Theorem 1 *If $\phi : [c, d] \to [a, b]$ is a parameter transform and $\gamma_1 : [a, b] \to \mathbb{C}$ an integration path, then $\gamma_2 = \gamma_1 \circ \phi : [c, d] \to \mathbb{C}$ is also a path of integration and for a function f continous on the image of γ_1 we have:*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz \tag{16}$$

Definition 1.5 *For a continous function f on an open set $E \subset \mathbb{C}$ we call $F : E \to \mathbb{C}$ the antiderivative of f, if F is holomorphic on E and $F' = f$. f is said to have local antiderivatives, if for each $z_0 \in E$ there exists a neighbourhood $G \subset E$ of z_0, so that there exists a holomorphic F with $F' = f$ on G.*

1.3 Holomorphic functions and the Cauchy integral theorem

Definition 1.6 *A complex function f is said to be analytic at a point z_0 if there exists a neighbourhood G of z_0, for example an open circle, so that f has a powerseries expansion, which converges for some radius R. f is said to be analytic on an open set, if it is analytic on every point of this set.*

Theorem 2 *For a continous function* f *on a domain*E *these are equivalent:*

1. f *is holomorphic*
2. f *has lacal antiderivatives*
3. f *is differentiable in the real sense and obeys the CRDEs*
4. f *is analytic*

holomorphic functions are arbitrarily often differentiable: outstanding property!

holomorphic functions are completly defined by only one point (all derivatives)

the values of the function at countable many points, which cluster at one point, define a holomorphic function completly

a theorem of Riemann:

$$f(z) \cdot \frac{z-c}{z-c} = f(z) \tag{17}$$

Theorem 3 *(CIT) Let E be a convex domain and $Z \subset E$ a set of points without cluster points. If $f : E \to \mathbb{C}$ is continous and holomorphic on $E \setminus Z$ then for every closed (meaning $\gamma(a) = \gamma(b)$) path of integration $\gamma : [a, b] \to \mathbb{C}$ the following holds:*

$$\int_\gamma f(z)dz = 0 \tag{18}$$

1.4 Cauchy integral formula and residue calculus

Theorem 4 *CIF Let U be an open disc in a domain E with $\overline{U} \subset E$ and we denote a positivly oriented boundary of U with ∂U Let $f : E \to \mathbb{C}$ be holomorphic, then the following statement hold:*

$$\forall z \in U: \quad f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{19}$$

$$\forall z \in U: \quad f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{20}$$

$$\forall z \in E \setminus \overline{U}: \quad 0 = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{21}$$

Laurent series:

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{22}$$

More or less every function is expressible as Laurent series.

Let f only have isolated singularities;

Definition 1.7 *The coefficient at $n = -1$ of a Laurent series of a holomorphic function f with isolated singularities of type (22) is called the residue at z_0:* $\underset{z_0}{\mathrm{Res}}(f) = a_{-1}$

There exists an ϵ such that:

$$\underset{z_0}{\mathrm{Res}}(f) = \frac{1}{2\pi i} \int_{\partial U_\epsilon(z_0)} f(z) dz \tag{23}$$

Theorem 5 *(Residue theorem) Let E be an open set and U an open disc with $E \subset U$. Let, for some $n \in \mathbb{N}_0$, f be holomorphic on $E \setminus \{z_1, \dots, z_n\}$. Then*

$$\frac{1}{2\pi i} \int_{\partial U} f(z) dz = \sum_{k=0}^{n} \underset{z_k}{\mathrm{Res}}(f) \tag{24}$$

2 Other mathematical formulas

2.1 The Gamma function

Generalisation of the factorial to complex numbers

$$\Gamma(s) := \int_0^\infty e^{-t} t^{x-1} dt \tag{25}$$

It satisfies the relations:

$$\forall n \in \mathbb{N}_0 : \Gamma(n) = (n-1)! \tag{26}$$

$$\forall x \in \mathbb{C} \setminus \mathbb{N}_0 : \Gamma(x+1) = x\Gamma(x) \tag{27}$$

A direct consequence is:

$$\prod_{i=0}^{n}(s-i) = \frac{\Gamma(s+1)}{\Gamma(s-n)} \tag{28}$$

The Gamma function is holomorphic on its domain $\mathbb{C} \setminus -\mathbb{N}_0$; at the negativ integers it diverges.

Stirling formula:

$$\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}(1+O(\frac{1}{n})) \tag{29}$$

The standard estimate:

$$\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} = n^{\alpha}(1+O(\frac{1}{n})) \tag{30}$$

γ is Eulers constant:

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i}-\ln(n)\right)=:\gamma\approx 0.577 \tag{31}$$

$$\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma-\frac{1}{x}-\sum_{k=1}^{\infty}\left(\frac{1}{x+k}-\frac{1}{k}\right) \tag{32}$$

The Beta function:

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{33}$$

2.2 Zeta functions and modified Bell polynomials

Incomplete Hurwitz ζ function:

$$\zeta_n(r,\beta) = \sum_{i=0}^{n-1} \frac{1}{(i+\beta)^r} \tag{34}$$

$\zeta_n(r,1)$ define the generalised harmonic numbers $\zeta_n(r)$ and their limit ($n \to \infty$) is the famous Riemann ζ function.

Using (32):

$$\zeta_{n+1}(1,\beta) = \ln(n) - \frac{\Gamma'(\beta)}{\Gamma(\beta)} + O(\frac{1}{n}) \tag{35}$$

Modified Bell polynomials $\mathbf{L_m} = \mathbf{L_m}(x_1, x_2, \ldots, x_m)$

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) = 1 + \sum_{m=1}^{\infty} \mathbf{L_m} t^m \tag{36}$$

$$\mathbf{L_m}(x_1, x_2, \ldots) = \sum_{1m_1+2m_2+\ldots=m} \frac{1}{m_1! m_2! \ldots} \left(\frac{x_1}{1}\right)^{m_1} \left(\frac{x_2}{2}\right)^{m_2} \ldots \tag{37}$$

The first polynomials are

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) = 1 + x_1 t + \left(\frac{x_2}{2} + \frac{x_1^2}{2}\right) t^2 + \left(\frac{x_3}{3} + \frac{x_1 x_2}{2} + \frac{x_1^3}{6}\right) t^3 +$$
$$\left(\frac{x_4}{4} + \frac{x_1 x_3}{3} + \frac{x_2^2}{8} + \frac{x_2 x_1^2}{4} + \frac{x_1^4}{24}\right) t^4 + \ldots \tag{38}$$

3 Motivation and basic integrals

Generalised diffrences for a sequence $\{f_k\}_{k\in\mathbb{N}_0}$:

$$\Delta f_n = f_{n+1} - f_n \quad \Delta^n f_0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f_k = (-1)^n D_n[f] \tag{39}$$

$$|D_n[f]| \leq 2^n \max_{0\leq k\leq n} |f_n| \tag{40}$$

In application often there are polynomial bounds – this phenomenon is called exponential cancelation.

Exponential generating:

$$f(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} \tag{41}$$

Poisson generating:

$$\hat{f}(z) = \sum_{n=0}^{\infty} f_n e^{-z} \frac{z^n}{n!} \tag{42}$$

Holomorphic function $\phi(x)$ interpolates the sequence f_n: $\forall k \in \mathbb{N}_0 : f_k = \phi(k)$.

Lemma 3.1 *Let ϕ be a holomorphic function in a domain that contains the half-line $[n_0, \infty[$ and $\mathcal{C}$ is a positivly oriented closed path in the domain of ϕ, which encircles $[n_0, n]$ and does not include any of the integers $0, 1, \dots, n_0 - 1$ nor a point, where ϕ is not holomorphic. Then the following holds*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = \frac{(-1)^n}{2\pi i} \int_{\mathcal{C}} \phi(s) \frac{n!}{s(s-1)\dots(s-n)} ds \qquad (43)$$

Remark: The kernel can also be rewriten as

$$\frac{n!}{s(s-1)\dots(s-n)} = \frac{\Gamma(n+1)\Gamma(s-n)}{\Gamma(s+1)} = B(n+1, s+1) = (-1)^{n-1} B(n+1, -s) \qquad (44)$$

Proof Apply the residue theorem (24).

Consider the poles $n_0, n_0 + 1, \ldots, n$

Let k be such an integer; use the CIF (19):

$$\operatorname*{Res}_{s=k} \phi(s) \frac{n!}{s(s-1)\ldots(s-n)} =$$

$$\operatorname*{Res}_{s=k} \frac{1}{s-k} \left(\phi(s) \frac{n!}{s(s-1)\ldots(s-k+1)(s-k-1)\ldots(s-n)} \right) =$$

$$\phi(k) \frac{(-1)^{n-k} n!}{k!(n-k)!} \tag{45}$$

Simple summation while taking the sign into account yields the proposition. $\square$

Definition 3.2 *A function ϕ in an unbounded domain Ω is said to have polynomial growth, if for some r the formula $|\phi(s)| = O(|s|^r)$ holds as $s \to \infty$ in Ω. r is called the degree of ϕ*

If the function in (43) is of polynomial growth in the half-plain $\Re(s) > n_0 - \epsilon$ for some $\epsilon > 0$ and n is sufficiently large, then we have for some $n_0 > c > \max\{n_0 - \epsilon, n_0 - 1\}$ the representation

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = -\frac{(-1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) \frac{n!}{s(s-1)\dots(s-n)} ds \tag{46}$$

4 Integrals of functions with poles and representation of sumes

4.1 Rational functions

Theorem 6 *Let ϕ be a rational function holomorphic in a domain that contains the half-line $[n_0, \infty[$. If n is big enough we have*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = -(-1)^n \sum_{s} \operatorname{Res}_{s} \left(\phi(s) \frac{n!}{s(s-1)\dots(s-n)} \right) \tag{47}$$

where the sum is taken over all poles of ϕ and over $0, 1, \dots, n_0 - 1$

Proof

Use Lemma 3.1

When $R \to \infty$ and $n > r$ the integral on the right side of (43) tends to 0

Apply the residue theorem (24) to (47) $\square$

Lemma 4.1 *When $\alpha \in \mathbb{C} \setminus \mathbb{N}_0$, then*

$$I_n(\alpha) = (-1)^n n! \operatorname*{Res}_{s=\alpha} \left(\frac{1}{(s-\alpha)^r} \frac{1}{s(s-1)\dots(s-n)} \right) \tag{48}$$

has the following asymptotic

$$I_n(\alpha) = -\Gamma(-\alpha) n^\alpha \frac{(\ln n)^{r-1}}{(r-1)!} \left(1 + O\left(\frac{1}{\ln n} \right) \right) \tag{49}$$

In the following we use the symbol $\langle \phi(s) \rangle_{k,\alpha}$ to denote the kth coefficient in the Laurent series at a certain point $\alpha \in \mathbb{C}$.

Proof

$$I_n(\alpha) = -n!\left\langle (s-\alpha)^r \frac{1}{(-s)(1-s)\dots(n-s)} \right\rangle_{-1,\alpha} =$$

$$-n!\left\langle \frac{1}{(-s)(1-s)\dots(n-s)} \right\rangle_{r-1,\alpha} =$$

$$-n!\left\langle \frac{1}{(-\alpha-s)(1-\alpha-s)\dots(n-\alpha-s)} \right\rangle_{r-1,0} =$$

$$-n!\left\langle \exp\left(-\ln\left(\prod_{j=0}^{n}(j-\alpha-s)\right)\right) \right\rangle_{r-1,0} =$$

$$-n!\left\langle \exp\left(-\sum_{j=0}^{n}\ln(j-\alpha-s)\right) \right\rangle_{r-1,0} = \quad (50)$$

$$\ln(j-\alpha-x) = \ln\left((j-\alpha)\left(1+\tfrac{-x}{j-\alpha}\right)\right) = \ln(j-\alpha) + \ln\left(1+\tfrac{-x}{j-\alpha}\right)$$

series expansion:

$$\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m+1} \tfrac{x^m}{m}$$

$$
\begin{aligned}
&= -n!\exp\left(-\sum_{j=0}^{n}\ln(j-\alpha)\right)\left\langle \exp\left(\sum_{j=0}^{n}\left(\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{s}{j-\alpha}\right)^{m}\right)\right)\right\rangle_{r-1,0} = \\
&-n!\frac{1}{(-\alpha)(1-\alpha)\dots(n-\alpha)}\left\langle \exp\left(\sum_{m=1}^{\infty}\left(\sum_{j=0}^{n}\frac{1}{(j-\alpha)^{m}}\frac{s^{m}}{m}\right)\right)\right\rangle_{r-1,0} \overset{(34)}{=} \\
&-n!\frac{1}{(-\alpha)(1-\alpha)\dots(n-\alpha)}\left\langle \exp\left(\sum_{m=1}^{\infty}\zeta_{n+1}(m,-\alpha)\frac{s^{m}}{m}\right)\right\rangle_{r-1,0} \overset{(37)}{=} \\
&-\frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)}\mathbf{L_{r\text{-}1}}(\zeta_{n+1}(1,-\alpha),\zeta_{n+1}(2,-\alpha),\dots\zeta_{n+1}(r-1,-\alpha)) \overset{(30)}{=} \\
&-\frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)}\mathbf{L_{r\text{-}1}}(\ln n-\frac{\Gamma'(-\alpha)}{\Gamma(-\alpha)}+O(1/n),\zeta_{n+1}(2,-\alpha),\dots\zeta_{n+1}(r-1,-\alpha)) =
\end{aligned}
\tag{51}
$$

$\zeta_n(r,\beta) = O(1)$ for $n \to \infty$ and $r \in \mathbb{N} \setminus \{1\}$

beside the first coefficient of $\mathbf{L_m}$ all coefficent are of degree smaller than m

$$= -\frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)} \frac{1}{(r-1)!} (\ln n - \frac{\Gamma'(-\alpha)}{\Gamma(-\alpha)} + O(1/n))^{r-1} =$$

$$-\frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)} \frac{(\ln n)^{r-1}}{(r-1)!} \cdot \left(1 + O\left(\frac{1}{\ln n}\right)\right) \overset{(35)}{=}$$

$$-\Gamma(-\alpha)n^{\alpha}\left(1 - O\left(\frac{1}{n}\right)\right) \frac{(\ln n)^{r-1}}{(r-1)!} \cdot \left(1 + O\left(\frac{1}{\ln n}\right)\right) =$$

$$-\Gamma(-\alpha)n^{\alpha} \frac{(\ln n)^{r-1}}{(r-1)!} \cdot \left(1 + O\left(\frac{1}{\ln n}\right)\right) \tag{52}$$

$\square$

Example: analyse for an $m \in \mathbb{N}$ the asymtotic growth of the sum

$$S_n(m) = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^m} \tag{53}$$

Use the function $\phi(s) = \frac{1}{s^m}$ to interpolate the sequence and set $n_0 = 1$. Only one pole is not in $\{n_0, n_0 + 1, \ldots, n\}$: 0 with order $m + 1$. We have to modify our calculations yielding (4.1), since the pole is in $\mathbb{N}_0$:

$$\begin{aligned} S_n(m) &= -\operatorname*{Res}_{s=0}\left(\frac{1}{s^{m+1}} \frac{n}{s-n} \frac{n-1}{s-n+1} \cdots \frac{2}{s+2} \frac{1}{s+1}\right) = \\ &-\operatorname*{Res}_{s=0}\left(\frac{1}{s^{m+1}} \left(\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\cdots\left(1-\frac{s}{n}\right)\right)^{-1}\right) = \\ &-\left\langle \left(\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\cdots\left(1-\frac{s}{n}\right)\right)^{-1} \right\rangle_{m,0} \end{aligned} \tag{54}$$

Similar to the calculations taken out in (50) and (51) with the generalised harmonic numbers $\zeta_n(k)$ this is

$$-\left\langle \exp\left(\sum_{k=1}^{\infty} \zeta_n(k) \frac{s^k}{k}\right) \right\rangle_{m,0} \tag{55}$$

Now we can use once again the modified Bell polynomials and the facts, that $\zeta_n(k) = \zeta(k) + O\left(\frac{1}{n^{k-1}}\right)$ for $k \geq 2$ and $\zeta_n(1) = \ln(n) + \gamma + O(1/n)$, which follows from (35) with $\beta = 1$ and $\Gamma'(1) = \gamma\Gamma(1)$ to get:

$$-S_n(m) = \sum_{1m_1+2m_2+\ldots=m} \frac{1}{m_1!m_2!\ldots} \left(\frac{\zeta_n(1)}{1}\right)^{m_1} \left(\frac{\zeta_n(2)}{2}\right)^{m_2} \left(\frac{\zeta_n(3)}{3}\right)^{m_3} \ldots =$$

$$\left(1 + O\left(\frac{1}{n}\right)\right) \sum_{1m_1+2m_2+\ldots=m} \frac{1}{m_1!m_2!\ldots} \left(\ln(n) + \gamma + O\left(\frac{1}{n}\right)\right)^{m_1}$$

$$\left(\frac{\zeta(2)}{2}\right)^{m_2} \left(\frac{\zeta(3)}{3}\right)^{m_3} \ldots \tag{56}$$

The $\zeta(k)$ are constants; for a polynomial P_m of degree m:

$$-S_n(m) = P_m(\ln(n)) + O\left(\frac{(\ln(n))^m}{n}\right) \tag{57}$$

For example:

$$-S_n(1) = \ln(n) + \gamma + O\left(\frac{1}{n}\right) \tag{58}$$

$$-S_n(2) = \frac{1}{2}(\ln(n))^2 + \gamma\ln(n) + \frac{\gamma}{2} + \frac{\pi^2}{12} + O\left(\frac{\ln(n)}{n}\right) \tag{59}$$

Moreover for $m = 1$ we get the exact result

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} = -S_n(1) = \zeta_n(1) \tag{60}$$

Another example is the sequence

$$T_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k^2+1} \tag{61}$$

This sequence obeys the recurrence

$$T_0 = 1 \quad T_1 = \frac{1}{2} \quad T_n = \frac{n}{n^2+1}((2n-1)T_{n-1} - (n-1)T_{n-2}) \tag{62}$$

Interpolation: $\phi(s) = (1+s^2)^{-1} = \frac{1}{s-i} + \frac{1}{s+i}$

Applay (4.1), use trigonometric identities and the fact that $|\Gamma(z)| = |\Gamma(\overline{z})|$ to get

$$\Gamma(-i)n^{i}\left(1+O\left(\frac{1}{n}\right)\right)+\Gamma(i)n^{-i}\left(1+O\left(\frac{1}{n}\right)\right)=$$
$$(\Gamma(-i)e^{i\ln(n)}+\Gamma(i)e^{-i\ln(n)})\left(1+O\left(\frac{1}{n}\right)\right)=\rho\cdot\cos(\ln(n)+\theta)+o(1) \tag{63}$$

for some θ and $\rho = 2\,|\Gamma(i)| = 2\sqrt{\pi/\sinh(\pi)} \approx 1.04313.$

4.2 Meromorphic functions

Meromorphic functions are generalisations of rational function. Meromorphic functions are holomorphic on an certain domain except isolated singularities.

Theorem 7 *Let ϕ be a function holomorphic in a domain that contains the half-line $[n_0, \infty[$. If n is big enough we have*

1. *If ϕ is meromorphic on $\mathbb{C}$ and of polynomial growth, then*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = -(-1)^n \sum_s \operatorname*{Res}_s \left(\phi(s) \frac{n!}{s(s-1)\dots(s-n)} \right) \tag{64}$$

where the sum is taken over all poles of ϕ and over $0, 1, \dots, n_0 - 1$

2. *If ϕ is meromorphic on the half-plane defined by $\Re(s) \geq d$ for some $d < n_0$ and of polynomial growth in this set, then*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = -(-1)^n \sum_{s} \operatorname*{Res}_{s} \left(\phi(s) \frac{n!}{s(s-1)\dots(s-n)} \right) + O(n^d) \tag{65}$$

where the sum is taken over all poles but $n_0, n_0 + 1, \dots, n$

Proof Since in both cases the function is meromorphic, the number of poles is countable. So in the first case we can find positively oriented, concentric circles γ_j whose radii tend to ∞ and do not come across any pole. In the second case we can find for any $\epsilon > 0$ a $d < d' < d + \epsilon$, such that the pathes defined by $[d' - iR_j, d' + iR_j]$ and the half circle with center d' and radius R_j don't cross a pole and R_j tends to infinity. Since d' is arbitrarily close to d, we can assume $d = d'$. Of course, if the theorem is used in praxis, other types of paths can be used, when they have the essential properties stated and used here.

Now we integrate along these curves and get using the residue theorem similar to theorem 6 the results above. Since ϕ is of polynomial growth we can use the arguments at (46) to get the asymtotics of the integrals over the "infinite" paths. In the first case we get 0 for n big enough to overwhelm the polynomial growth. In the second case the $O(n^d)$ comes from the integral along the parallel to the imaginary axes, as this argument illustrates (It's not a proof – it's just a plausibility argument; we asume $|\phi(s)| = O(|s|^r)$ for an integer r):

$$\left|\frac{(-1)^n}{2\pi i}\int_{d-i\infty}^{d+i\infty}\phi(s)\frac{n!}{s(s-1)\dots(s-n)}ds\right| \overset{(14)}{\leq}$$

$$\frac{1}{2\pi}\int_{d-i\infty}^{d+i\infty}\left|\phi(s)\frac{n!}{s(s-1)\dots(s-n)}\right|ds \leq$$

$$\frac{1}{2\pi}\int_{d-i\infty}^{d+i\infty}|s|^r\left|\frac{n!}{s(s-1)\dots(s-n)}\right|ds =$$

$$\frac{1}{2\pi}\int_{d-i\infty}^{d+i\infty}\frac{|s|^r}{|s(s-1)\dots(s-r+1)\cdot(s-r)(s-r-1)|}\,|(n(n-1)\dots(n-d+1))|$$

$$\left|\frac{(n-d)(n-d-1)\dots(r+2-d)}{(n-s)(n-s-1)\dots(r+2-s)}\right|\,|(r+2-d)(r+1-d)(r-d)\dots2\cdot1|^{-1}\,ds \leq$$

$$O(n^d)\int_{d-i\infty}^{d+i\infty}\frac{1}{|s|^2}ds = O(n^d) \quad (66)$$

This rough approximation holds for $r+1-d<0$. $\square$

As an application of this theorem, we could analyse the recurence relation

$$f_n = a_n + 2\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^n} f_k \tag{67}$$

This recurrence appears in digital sorting and tries. Niccolai will do this in the next talk.

Another example is the sum $V_n = \sum_{k=1}^{n-1} \binom{n}{k} \frac{B_k}{2^k - 1}$, which arises in the analysis of Patricia tries. Since $B_k = 0$ for $k \in 2\mathbb{N} + 1$, the sign is not necessary, and using $B_k = -k\zeta(1-k)$ for $k \in 2\mathbb{N} \cup \{1\}$, we have to analyse the integral

$$V_n = \frac{(-1)^n}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{n!}{(s-1)(s-2)\dots(s-n)} \frac{\zeta(1-s)}{2^s - 1} ds \tag{68}$$

The path of integration is the infinite rectangular from $\frac{1}{2} - \infty$ to $\frac{1}{2} + \infty$ and from $n - \frac{3}{4} + \infty$ to $n - \frac{3}{4} - \infty$; but it can be shown that the integral of the second path is identical 0 for each n and so the rectangular can be extended to ∞ so this is a variant of theorem 7(2), where the residues are not yet evaluated.

Now we have to consider the double pole at 0 (from the ζ function and from $(2^s-1)^{-1}$) and all the simple poles at $\chi_k = 2\pi i k/\ln(2)$, which yields

$$V_n = \frac{\ln(n)}{\ln(2)} - \frac{1}{2} - \frac{1}{\ln(2)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \zeta(1-\chi_k)\Gamma(1-\chi_k)e^{2\pi i k \ln(n)/\ln(2)} + O(\sqrt{n}) \tag{69}$$

There are also some other examples, where Rice's integrals can be used succesfully; for example digital trees or quad trees used for multidimensional searching.

Finding interpolating functions:

$$A_n = \prod_{k=1}^{n} \alpha_k \Rightarrow A(s) = \prod_{k=1}^{\infty} \frac{\alpha(k)}{\alpha(k+s)}$$

$$A_n = \sum_{k=1}^{n} \alpha_k \Rightarrow A(s) = \sum_{k=1}^{\infty} (\alpha(k) - \alpha(k+s))$$

4.3 Functions with algebraic and logarithmic singularities

Theorem 8 *For any nonintegral* λ*, the sum*

$$S_n(\lambda) = \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^{-\lambda} \tag{70}$$

has an asymptotic expansion in descending powers of $\ln(n)$ *of the form*

$$-S_n(\lambda) \sim (\ln(n))^{\lambda} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma^{(j)}(1)}{j!\Gamma(1+\lambda-j)} \frac{1}{(\ln(n))^j} \tag{71}$$

Proof As seen in (54) we can represent the sum as an integral in an analoge way

$$S_n(\lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}} \omega_n(s) \frac{ds}{s^{\lambda+1}} \text{ with } \omega_n(s) = \left(\left(1 - \frac{s}{1}\right)\left(1 - \frac{s}{2}\right) \dots \left(1 - \frac{s}{n}\right)\right)^{-1} \tag{72}$$

Here $\mathcal{C}$ may be the vertical line $\Re(s) = \frac{1}{2}$ as in (46). Despite all, these integrals would be hard to evaluate; since we are only interested to encircle the poles (here only $1, 2, \dots, n$) we can deformate the path of integration as we like unless we don't encounter any new pole or the slash. For this reason we introduce the Hankel contour.

Hankel contour:

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4$$

$$\mathcal{C}_1 = \left\{ s \middle| |s| = R \wedge \left(|\Im(s)| \geq \frac{1}{\ln(n)} \vee \Re(s) > 0 \right) \right\}$$

$$\mathcal{C}_2 = \left\{ s \middle| s = \frac{i - t}{\ln(n)} \wedge t \geq 0 \wedge |s| \leq R \right\}$$

$$\mathcal{C}_3 = \left\{ s \middle| s = \frac{e^{\theta i}}{\ln(n)} \wedge \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right\}$$

$$\mathcal{C}_4 = \left\{ s \middle| s = \frac{-i - t}{\ln(n)} \wedge t \geq 0 \wedge |s| \leq R \right\}$$

Now we split the integral in three parts $S_n(\lambda) = J_1 + J_2 + J_3$; we will see that for the asymptotic J_1 and J_2 can be neglected.

1. Let J_1 be the part, that belongs to the outer "circle" $\mathcal{C}_1$. Of course $s^{-\lambda-1}$ is of polynomial growth, so we can once more use our estimate for polynomial growth, to see that J_1 is $O(R^{-n-\lambda})$ and in the end, when $R \to \infty$, we get $J_1 = 0$

2. Now we will estimate the parts of $\mathcal{C}_2$ and $\mathcal{C}_4$ with $\Re(s) < -\frac{1}{\sqrt{\ln(n)}} =: -t_0$.

   For simplification, we forget about the term $s^{-\lambda-1}$ and asume we integrate along the negative real axes till we reach $-t_0$[a]. This simplifications don't touch the character of our estimate, but make it way simpler.

[a]We asume $\lambda + 1 \geq 0$ and then it is quiet obvious, that we can neglect the powers of s for the asymptotic analysis, because they would make the asymptotic only smaller; but for example if λ are negative integers we get the Stirling numbers of second kind, which have normaly exponenetial growth – our derivation can't be applayed in this case

Additionaly, we mirror everything on the imaginary axes, to get:

$$\mu(n) := \int_{t_0}^{\infty} \frac{dt}{(1+\frac{t}{1})\dots(1+\frac{t}{n})} \tag{73}$$

As next step we split the new path into the intervalls $[t_0, 1]$, $\left[1, n^{1/3}\right]$ and $\left[n^{1/3}, \infty\right]$, to get the integrals $\mu_1(n)$, $\mu_2(n)$ and $\mu_3(n)$

(a)

$$\mu_1(n) = \int_{t_0}^{1} \frac{dt}{(1+t/1)\dots(1+t/n)} =$$

$$\int_{t_0}^{1} \frac{n!dt}{(1+t)(2+t)\dots(n+t)} = \int_{t_0}^{1} \frac{\Gamma(n+1)\Gamma(t+1)}{\Gamma(n+t+1)}dt \overset{(29)}{=}$$

$$\int_{t_0}^{1} \frac{\sqrt{2\pi}(n+1)^{n+1-1/2}e^{-n-1}\sqrt{2\pi}(t+1)^{t+1-1/2}e^{-t-1}}{\sqrt{2\pi}(n+t+1)^{n+t+1-1/2}e^{-n-t-1}} \left(1+O\left(\frac{1}{n}\right)\right) dt \leq$$

$$\int_{t_0}^{1} \frac{\sqrt{2\pi}(t+1)^{t+1-1/2}}{e(n+t+1)^{t}} \left(1+O\left(\frac{1}{n}\right)\right) dt =$$

$$O(1)\int_{t_0}^{1} \frac{(t+1)^{t+1-1/2}}{(n+t+1)^{t}}dt = O(1)\int_{t_0}^{1} \frac{1}{(n)^{t}}dt \quad (74)$$

The last few transformation can be carried out, because $t \in [t_0, 1]$, so is small compared to n and can be ignored or estimated by a constant respectivly.

$$O(1)\int_{t_0}^{1}\frac{1}{(n)^t}dt = O(1)\left(\frac{-1+n^{1-t_0}}{n\ln(n)}\right) =$$
$$O(e^{-t_0\ln(n)+\ln(\ln(n))}) = O(e^{-1/2\sqrt{\ln(n)}}) \quad (75)$$

(b) In the next part we use, that $1 \leq t$

$$\mu_2(n) = \int_1^{n^{1/3}}\frac{n!}{(1+t)\dots(n+t)}dt =$$
$$\int_1^{n^{1/3}} 1\cdot 2\cdot\frac{3}{2+t}\frac{4}{3+t}\cdots\frac{n}{n-1+t}\frac{1}{(1+t)(n+t)}dt \leq$$
$$\int_1^{n^{1/3}} O(1)\frac{1}{(1+t)(n+t)}dt = O\left(\frac{\ln(n^{1/3}+n)}{n-1}\right) = O(n^{-2/3}) \quad (76)$$

(c) Similar to the estimates done in the formula above we get, for n large enough

$$\mu_3(n) = \int_{n^{1/3}}^{\infty} \frac{1}{(1+t/1)\dots(1+t/n)} dt \leq$$
$$\int_{n^{1/3}}^{\infty} \frac{1}{e^{(1/2)t}} dt = O(e^{-(1/2)n^{1/3}}) \quad (77)$$

In the whole we have the result $J_2 = O(e^{(1/2)\sqrt{\ln(n)}})$, so it is of smaller order than any negativ power of $\ln(n)$

3. Now we have to estimate J_3; this is the integral along the contour of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$, for which $\Re(s) \geq -t_0$, denoted in the following by $\mathcal{C}^0$.
Again we use Stirlings formula (29) to get the asymptotic:

$$\omega_n(s) = n^s \Gamma(1-s)\left(1 + O\left(\frac{\ln(n)}{n}\right)\right) \quad (78)$$

Since s is very small on $\mathcal{C}^0$ the part with $O\left(\frac{\ln(n)}{n}\right)$ can be estimated easily by $O(e^{(1/2)\sqrt{\ln(n)}})$ and this yields

$$J_3 = J^0 + O(e^{(1/2)\sqrt{\ln(n)}}) \text{ with } J^0 = \frac{1}{2\pi i}\int_{\mathcal{C}^0} n^s \Gamma(1-s)\frac{ds}{s^{\lambda+1}} \quad (79)$$

Now we use the transformation $z = s\ln(n)$ with $\mathcal{D}^0$ is the image under this transform of $\mathcal{C}^0$ and have

$$J^0 = (\ln(n))^\lambda \frac{1}{2\pi i}\int_{\mathcal{D}^0} e^z \Gamma\left(1 - \frac{z}{\ln(n)}\right)\frac{dz}{z^{\lambda+1}} \quad (80)$$

By the transform we get $|z| = O(\sqrt{\ln(n)})$ on $\mathcal{D}^0$; this is the reason, we can expand the Gamma function around 1 in a power series and after changing summation and integration, we get

$$J^0 = (\ln(n))^\lambda \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma^{(m)}(1)}{m!} \frac{1}{(\ln(n))^m} \frac{1}{2\pi i} \int_{\mathcal{D}^0} e^z z^{m-\lambda-1} dz \tag{81}$$

Now we have to estimate the remaining integrals; this can be done by the so called Laplace method, where we extend the contour to $-\infty$ to get $\mathcal{L}$. We will only present the result here:

$$\frac{1}{2\pi i} \int_{\mathcal{L}} e^z z^{m-\lambda-1} dz = \frac{1}{\Gamma(1-m+\lambda)} \tag{82}$$

Now taking together all the parts we have proofed the theorem. $\square$

Example:

$$X_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{\sqrt{1+k^2}} \tag{83}$$

Local behvior $(s \pm i)^{-1/2}$ at the "problem points". Similar to (63):

for some ρ and θ_0

$$X_n = \rho\sqrt{\ln(n)}\cos(\ln(n)+\theta_0) + O((\ln(n))^{-1/2}) \tag{84}$$

Other examples and direct applications are

$$-S_n(-1/2) = \frac{1}{\sqrt{\pi \ln(n)}} - \frac{\gamma}{2\sqrt{\pi(\ln(n))^3}} + O((\ln(n))^{-5/2}) \qquad (85)$$

$$-S_n(1/2) = 2\sqrt{\frac{\ln(n)}{\pi}} + \frac{\gamma}{\sqrt{\pi \ln(n)}} + O((\ln(n))^{-3/2}) \qquad (86)$$

Theorem 9 *(without proof) For the logarithmic differences we have the asymptotic*

$$Y_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \ln(k) = \ln(\ln(n)) + \gamma + \frac{\gamma}{\ln(n)} - \frac{\pi^2 + 6\gamma^2}{12(\ln(n))^2} + O\left(\frac{1}{(\ln(n))^3}\right) \tag{87}$$

For entire functions for example:

$$Z_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} \tag{88}$$

which obeys the recurrence $Z_{n+2} = (2 - 2/n)Z_{n+1} + (1 - 1/n)Z_n$ can be extrapolated by the entire function $1/\Gamma(s)$

for some constants c and θ

$$Z_n = cn^{-1/4} \sin(2n^{1/2} + \theta) + o(n^{-1/4}) \tag{89}$$

We have seen many cases were havy use of complex analysis can resolve the asymptotics of recurrences, generalised diffrences and sums. We summaries this in the table below.

Some singularities and the introduced asymptotics in the corresponding difference

singularity	asymptotics
singularity of $\phi(s)$ at $s_0 = \sigma_0 + i\tau$	approximatly $n^{s_0} = n^{\sigma_0} e^{i\tau_0 \ln(n)}$
simple pole: $(s - s_0)^{-1}$	$-\Gamma(-s_0)n^{s_0}$
multiple pole: $(s - s_0)^{-r}$	$-\Gamma(-s_0)n^{s_0} \frac{(\ln(n))^{r-1}}{(r-1)!}$
algebraic singularity: $(s - s_0)^{\lambda}$	$-\Gamma(-s_0)n^{s_0} \frac{(\ln(n))^{-\lambda-1}}{\Gamma(-\lambda)}$
logarithmic singularity: $(s - s_0)^{\lambda}(\ln(s - s_0))^{r}$	$-\Gamma(-s_0)n^{s_0} \frac{(\ln(n))^{-\lambda-1}}{\Gamma(-\lambda)\ln(\ln(n))^{r}}$

5 Mellin transforms and Rice's integrals

The Mellin transform of a function and its inverse have the form

$$\phi(z) = \int_0^\infty t^{z-1} f(t) dt \qquad f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} \phi(z) dz \tag{90}$$

Take a closer look to the integral (46) and use assymptotic of $\omega_n(s)$ (78):

$$\frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \phi(s) \frac{(-1)^n n!}{s(s-1)\dots(s-n)} ds \approx \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \phi(s) \Gamma(-s) n^s ds \tag{91}$$

The Poisson-Mellin-Newton cycle:

Theorem 10 *The coefficients of a Poisson generating function are expressible as a Rice's integral of a Mellin transform of the Poisson generating function.*

$$\{f_n\} \xrightarrow{\text{Poisson GF}} \hat{f}(t) = \sum_{n=0}^{\infty} f_n e^{-t} \frac{t^n}{n!} \xrightarrow{\text{Mellin transform}}$$
$$\hat{f}^*(s) = \int_0^{\infty} \hat{f}(t) t^{s-1} dt \xrightarrow{\text{Rice's integral}} \{f_n\} \qquad (92)$$

Proof

$$\hat{f}^*(s) = \int_0^\infty \hat{f}(t) t^{s-1} dt = \sum_0^\infty \frac{f_n}{n!} \int_0^\infty e^{-t} t^{s+n-1} dt \overset{(25)}{=}$$

$$\Gamma(s) \left(f_0 + f_1 \frac{s}{1!} + f_2 \frac{s(s+1)}{2!} + \ldots \right) \tag{93}$$

By differencing:

$$f_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\hat{f}^*(-s)}{\Gamma(-s)} \tag{94}$$

But these differences are exactly of the Rice type, so we conclude the corresponding equations

$$f_n = \frac{(-1)^n}{2\pi i} \int_{\mathcal{C}} \left(\frac{\hat{f}^*(s)}{\Gamma(-s)} \frac{n!}{s(s-1)\dots(s-n)} \right) ds \tag{95}$$

$$\hat{f}^*(s) = \int_0^\infty \left(e^{-t} \sum_{n=0}^{\infty} f_n \frac{t^n}{n!} \right) t^{s-1} dt \tag{96}$$

This are the relations, which were stated. □

6 Summary

Despite complex analysis seems to be part of pure mathematics, it can be applied for finding asymptotics of solutions of difference equations and generalised diffrences, which are needed in the analysis of algorithms. By the method of Rice's integrals we can tackle the average case analysis of tries, digital search trees, multidimensional searching and other datastructures and algorithms of great practical use. In most cases a detailed calculation of the asymptotics is in fact much to complex, but you can get a first estimate of the growth by comparing the problem with the examples outlined here and the table given at the end of the section before the last.[a]

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[a]spezial thanx 2:
Olga, Ilja, Dimitri
Flajolet, Sedgwick, Rice, Heise, Maass and
Rage, Nirvana, Metallica, Hammerfall, Queens of the Stone Age, Marilyn Manson . . .